

Strong scale dependent bispectrum in the Starobinsky model of inflation

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We compute analytically the dominant contribution to the tree-level bispectrum in the Starobinsky model of inflation. In this model, the potential is vacuum energy dominated but contains a subdominant linear term which changes the slope abruptly at a point. We show that on large scales compared with the transition scale k_0 and in the equilateral limit the analogue of the non-linearity parameter scales as $(k/k_0)^2$, that is its amplitude decays for larger and larger scales until it becomes subdominant with respect to the usual slow-roll suppressed corrections. On small scales we show that the non-linearity parameter oscillates with angular frequency given by $3/k_0$ and its amplitude grows linearly towards smaller scales and can be large depending on the model parameters. We also compare our results with previous results in the literature.

I. INTRODUCTION

The idea of inflation in the very early universe [1–6] is a very compelling solution to the problems of the Big Bang theory. However one should keep in mind that there may be other viable alternative explanations. Other successful predictions [7] of the simplest inflationary models are the nearly scale invariant, mostly of adiabatic origin and nearly Gaussian primordial curvature perturbation. This primordial perturbation is generated due to quantum vacuum fluctuations of the scalar field driving inflation and will at a much later stage in the evolution of the universe give origin to the cosmic microwave background (CMB) radiation anisotropies and the large-scale structure of galaxies in the universe.

While the simplest models are appealing due to their simplicity and fewer ingredients they are not always realistic and often more realistic models introduce several fields, non-trivial potentials and kinetic terms, etc.. All these models have to pass the observational constraints imposed by current data on the cosmological background as well as on the power spectrum of the perturbations. In many cases the lack of other observables renders many models observationally indistinguishable. However some models have other observables like non-Gaussianity (i.e. non-linearity of the perturbations) and these can be and are being used to further constrain and distinguish models.

Non-Gaussianity of the primordial curvature perturbation has not been observed, however the CMB data still allows a small non-Gaussianity ($\sim 0.1\%$) and there have been some claims of detections or hints of its presence [8–11]. Despite this small value, efforts are underway (e.g. ESA's Planck satellite [12]) to measure the CMB anisotropies to a precision with which this small amount would be detected to many standard deviations.

It also turns out that the simplest models of inflation, i.e. the ones driven by a single scalar field with standard kinetic term, with a potential satisfying the slow-roll conditions and if one assumes the standard initial conditions for the quantum vacuum fluctuations, predict a small and unobservable level of primordial non-linearity [13–18]. Therefore only models that can produce large non-Gaussianity can be constrained if no detection is made. One of such models put forward by Starobinsky in 1992 [19] will be the focus of this work. Many models which break at least one of the above conditions have been proposed and studied recently. An incomplete list is: for models that use several fields [20–51], for models with non-canonical kinetic terms [52–63], models with temporary violations of the slow-roll conditions or small departures of the initial vacuum state from the standard Bunch-Davies vacuum [64–73]. For reviews on the theory of non-Gaussian perturbations see [74–78].

The initial motivation of the Starobinsky model [19] that breaks temporarily the slow-roll approximation was to explain an observation that the correlation function of galaxies seemed to require more power on large scales than at smaller length scales within the paradigm of a Einstein-de-Sitter universe (a spatially flat universe with cold dark matter particles). It was also used to explain the possible nonconservation (enhancement) of the curvature perturbation on superhorizon scales due to not-fast-enough decay of the decaying mode [79]. More recently similar models [19, 80–85] were revived and proposed because it was realized that they can provide better fits to the CMB anisotropies power spectrum than the primordial power-law model [86–91]. Another reason for this interest is that it was realized that such models have very distinctive bispectrum signatures like for instance non-linearity parameters that are strongly scale-dependent. A particularly appealing feature of the Starobinsky model, a vacuum energy

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dominated potential added with a linear term that changes slope abruptly at a point, is that despite breaking a slow-roll condition it admits an analytical treatment of the background and linear perturbations equations as well as the non-Gaussianity as was shown in [70, 92] and in the present paper.

The main goal of this work is to compute analytically the leading-order bispectrum for the Starobinsky model. The bispectrum on large scales was first computed in [70] using a next-to-leading order gradient expansion formulation and more recently on all scales in the equilateral limit in [92], we compare the results of our calculation with these works.

This paper is divided in four sections and an appendix. In Section II, we describe the model and the analytical solutions for the background and linear perturbations are briefly reviewed. In Section III, we compute the leading-order contribution to the bispectrum in the equilateral limit. In the second subsection we shall discuss and compare our results with previous attempts in the literature. In the last subsection we elaborate on the size and shape of the analogue of the non-linearity parameter in this model. Section IV is devoted to conclusions. In Appendix A we present the lengthy result for the dominant contribution to the bispectrum that is valid for any triangular configuration (not only in the equilateral limit) of the three momenta on which the bispectrum depends.

We work in units where the reduced Planck constant \hbar and the speed of light c are $\hbar = c = 1$ and the reduced Planck mass is $M_{Pl} = (8\pi G)^{-1/2}$, where G is Newton's gravitational constant.

II. THE MODEL

In this section we introduce the model under consideration and review the analytical solutions for the background and the perturbation equations. All the results of this section can be found in Refs. [19, 92], here we only briefly review the main results that will be needed in the computation of the bispectrum of the next section. We use a somewhat different notation at points.

The Starobinsky model consists of a canonical scalar field minimally coupled to Einstein gravity with a potential given by

$$V(\phi) = \begin{cases} V_0 + A_+ (\phi - \phi_0) & : \phi > \phi_0, \\ V_0 + A_- (\phi - \phi_0) & : \phi < \phi_0, \end{cases} \quad (1)$$

where ϕ is the scalar field value, V_0 , A_+ and A_- are model parameters that are assumed to be positive. The sharp change in the slope occurs at $\phi = \phi_0$ and ϕ_0 is another free parameter of the model. We also assume that $|V - V_0|/V_0 \ll 1$, which means that the potential is vacuum energy dominated.

In the present case we are interested in a Friedmann-Lemaître-Robertson-Walker flat, homogeneous and isotropic background spacetime given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (2)$$

where t denotes cosmic time and $a(t)$ is the cosmic scale factor. The relevant equations of motion are the Friedmann and Klein-Gordon equations given respectively by

$$H^2 = \frac{1}{3M_{Pl}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right), \quad \ddot{\phi}(t) + 3H\dot{\phi}(t) + \frac{dV}{d\phi} = 0, \quad (3)$$

where the Hubble rate is defined as usual $H = \dot{a}/a$ and dot denotes the derivative with respect to t . The slow-roll parameters are defined by

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2M_{Pl}^2}, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H} = 2 \left(\frac{\ddot{\phi}}{\dot{\phi}H} + \epsilon \right), \quad (4)$$

and will be assumed to be small before the transition. After the transition ϵ will continue to be small but η will become large for some time before the slow-roll regime is again recovered.

With the above mentioned assumption of a vacuum dominated potential the dynamics for the scale factor becomes particularly simple and is described by

$$H^2 \approx \frac{V_0}{3M_{Pl}^2}, \quad (5)$$

which has a solution $a(t) = \exp(H_0 t)$, where $H_0 = \sqrt{V_0/(3M_{Pl}^2)}$ and we chose $a(t=0) = 1$. In conformal time, denoted by τ , the previous solution is $a(\tau) = -1/(H_0 \tau)$ and one has $-H_0 t = \ln(-H_0 \tau)$. Before the transition the Klein-Gordon equation can also be easily solved to find [19, 92]

$$\phi_+ \simeq \phi_i + \frac{A_+ M_{Pl}^2}{V_0} \ln(-H_0 \tau), \quad \frac{d\phi_+}{d\tau} \simeq -aH \frac{A_+ M_{Pl}^2}{V_0}, \quad (6)$$

where ϕ_i is the initial value of the field. The subscript $+$ denotes quantities before the transition and the subscript $-$ denotes quantities after the transition.

After the transition, the Klein-Gordon can also be solved despite the fact that the slow-roll parameter η_- becomes temporarily large (however ϵ_- is small), one finds [92]

$$\phi_- \simeq \phi_0 + \frac{\Delta A}{9H_0^2} \left(1 - \left(\frac{\tau}{\tau_0} \right)^3 \right) - \frac{A_-}{3H_0^2} \ln \left(\frac{\tau_0}{\tau} \right), \quad (7)$$

where $\Delta A = A_- - A_+$ and τ_0 is the transition conformal time. The slow-roll parameter ϵ before and after the transition can be written respectively as

$$\epsilon_+ \simeq \frac{A_+^2}{18M_{Pl}^2 H_0^4}, \quad \epsilon_- \simeq \frac{A_-^2}{18M_{Pl}^2 H_0^4} (1 - \rho^3 \tau^3)^2, \quad (8)$$

where $\rho^3 \equiv -(\Delta A/A_-)k_0^3$ and k_0 is the transition scale given by $k_0 = a(\tau_0)H_0 = -1/\tau_0$. It is worth noting that before the transition $\eta_+ \approx 4\epsilon_+$ and at late times (much after the transition) we have again $\eta_- \approx 4\epsilon_-$. For a transient period after τ_0 , η_- can become large and it is well approximated by

$$\eta_- \simeq \frac{6\rho^3 \tau^3}{1 - \rho^3 \tau^3}. \quad (9)$$

In Ref. [92] it has been shown that the previous expressions for the slow-roll parameters agree very well with the exact numerical solutions. For the plots of the next section we choose the same parameters as the main choice of Ref. [92], i.e. $V_0 = 2.37 \times 10^{-12} M_{Pl}^4$, $A_+ = 3.35 \times 10^{-14} M_{Pl}^3$, $A_- = 7.26 \times 10^{-15} M_{Pl}^3$, this gives $\Delta A/A_- \approx -3.61$. This choice satisfies the constraint on the normalization of the primordial power spectrum imposed by COBE observations.

We now briefly describe the analytical approximation for the mode functions, all the details of the derivation can be found in [19, 92].

We work in the comoving gauge, so that the background value of ϕ , i.e. ϕ_0 , determines when the transition happens even including perturbations. In this gauge the three-dimensional metric is perturbed as

$$h_{ij} = a^2 e^{2\mathcal{R}} \delta_{ij}, \quad (10)$$

where \mathcal{R} denotes the comoving curvature perturbation and tensor perturbations were neglected because they do not contribute for the tree-level scalar bispectrum.

The Mukhanov-Sasaki equation in Fourier space is

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0, \quad (11)$$

where $z = a\sqrt{2\epsilon}$, prime denotes derivative with respect to the conformal time τ and the Fourier transform is defined by

$$\mathcal{R}_{\mathbf{k}} \equiv \mathcal{R}(\tau, \mathbf{k}) = \int d^3x \mathcal{R}(\tau, \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (12)$$

The standard quantization procedure in quantum field theory is to promote \mathcal{R} to an operator that is expanded in terms of creation and annihilation operators and mode functions as

$$\hat{\mathcal{R}}(\tau, \mathbf{k}) = \mathcal{R}(\tau, \mathbf{k}) \hat{a}(\mathbf{k}) + \mathcal{R}^*(\tau, -\mathbf{k}) \hat{a}^\dagger(-\mathbf{k}), \quad (13)$$

where the annihilation operator \hat{a} and the creation operator \hat{a}^\dagger satisfy the usual commutation relation $[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. The power spectrum of the curvature perturbation is given by

$$2\pi P_{\mathcal{R}}^{1/2}(k) = \sqrt{2k^3} |\mathcal{R}_k(\tau_f)|, \quad (14)$$

where τ_f is the end of inflation time.

Before the transition, the slow-roll approximation is valid and the mode function solution of Eq. (11) that reduces to the usual Minkowski result on small scales is the well known leading order slow-roll expression [19, 92],

$$\mathcal{R}_+(\tau, \mathbf{k}) = -\frac{3H_0^3}{A_+} \frac{1}{\sqrt{2k^3}} (k\tau - i) e^{-ik\tau}, \quad (15)$$

and one can take the derivative to find

$$\mathcal{R}'_+(\tau, \mathbf{k}) = -\frac{3H_0^3}{A_+} \frac{1}{\sqrt{2k^3}} (-ik^2\tau) e^{-ik\tau}. \quad (16)$$

This last equation is valid at leading order in slow-roll and will be used in the computation of the dominant contribution to the bispectrum. After the transition an analytical approximation can also be found, see [19, 92] for all the details. Here we just present the final result which is given by

$$\mathcal{R}_-(\tau, \mathbf{k}) = \frac{iH_0\alpha_k}{2M_{Pl}\sqrt{k^3\epsilon_-}} (1 + ik\tau) e^{-ik\tau} - \frac{iH_0\beta_k}{2M_{Pl}\sqrt{k^3\epsilon_-}} (1 - ik\tau) e^{ik\tau}, \quad (17)$$

where the Bogoliubov coefficients are

$$\alpha_k = 1 + \frac{3i\Delta A}{2A_+} \frac{k_0}{k} \left(1 + \frac{k_0^2}{k^2}\right), \quad \beta_k = -\frac{3i\Delta A}{2A_+} \frac{k_0}{k} \left(1 + i\frac{k_0}{k}\right)^2 e^{2ik/k_0}. \quad (18)$$

The most important feature of the above solution is the presence of the negative frequency mode which introduces oscillations in the power spectrum. For the derivative one finds

$$\begin{aligned} \mathcal{R}'_-(\tau, \mathbf{k}) = & \frac{iH_0\alpha_k}{2M_{Pl}\sqrt{k^3\epsilon_-}} \left[-\mathcal{H}\epsilon_- (1 + ik\tau) - \frac{\mathcal{H}\eta_-}{2} (1 + ik\tau) + k^2\tau \right] e^{-ik\tau} \\ & - \frac{iH_0\beta_k}{2M_{Pl}\sqrt{k^3\epsilon_-}} \left[-\mathcal{H}\epsilon_- (1 - ik\tau) - \frac{\mathcal{H}\eta_-}{2} (1 - ik\tau) + k^2\tau \right] e^{ik\tau}, \end{aligned} \quad (19)$$

where $\mathcal{H} = aH$ is the conformal Hubble parameter. This expression will also be used in the computation of the dominant bispectrum contribution in the next section. Because we are only interested in the dominant contribution we neglect the first term inside the square brackets because it is proportional to ϵ which is always small in the present model.

III. THE EQUILATERAL BISPECTRUM

In this section we compute the leading-order dominant contribution to the bispectrum in the equilateral limit. After that we compare our result with previous results in the literature. Finally we discuss the shape and size of the analogue of the non-linearity parameter in this model. We leave for Appendix A the lengthy result of the dominant contribution to the bispectrum that is valid for any triangular configuration of the momenta.

A. Calculation using the two vertices action

In order to compute the tree-level scalar bispectrum, we use the in-in formalism [93, 94]. For this we need to find the cubic-order interaction Hamiltonian, see e.g. Ref. [74] for a review about this procedure.

The third order action after ignoring total derivative terms can be found for instance in [13, 55, 95, 96]. It reads (here we follow the notation of [96])

$$\begin{aligned} S_3 = & M_{Pl}^2 \int dt d^3x \left[a^3 \epsilon^2 \mathcal{R} \dot{\mathcal{R}}^2 + a \epsilon^2 \mathcal{R} (\partial \mathcal{R})^2 - 2a \epsilon \dot{\mathcal{R}} (\partial \mathcal{R}) (\partial \chi) \right. \\ & \left. + \frac{a^3 \epsilon}{2} \frac{d\eta}{dt} \mathcal{R}^2 \dot{\mathcal{R}} + \frac{\epsilon}{2a} (\partial \mathcal{R}) (\partial \chi) \partial^2 \chi + \frac{\epsilon}{4a} (\partial^2 \mathcal{R}) (\partial \chi)^2 + 2 \left(\frac{\eta}{4} \mathcal{R}^2 + \tilde{f}(\mathcal{R}) \right) \frac{\delta L}{\delta \mathcal{R}} \Big|_1 \right], \end{aligned} \quad (20)$$

where we have defined

$$\chi = a^2 \epsilon \partial^{-2} \dot{\mathcal{R}}, \quad \left. \frac{\delta L}{\delta \mathcal{R}} \right|_1 = a \left(\frac{d\partial^2 \chi}{dt} + H \partial^2 \chi - \epsilon \partial^2 \mathcal{R} \right), \quad (21)$$

$$\tilde{f}(\mathcal{R}) = \frac{1}{H} \mathcal{R} \dot{\mathcal{R}} + \frac{1}{4a^2 H^2} [-(\partial \mathcal{R})(\partial \mathcal{R}) + \partial^{-2}(\partial_i \partial_j (\partial_i \mathcal{R} \partial_j \mathcal{R}))] + \frac{1}{2a^2 H} [(\partial \mathcal{R})(\partial \chi) - \partial^{-2}(\partial_i \partial_j (\partial_i \mathcal{R} \partial_j \chi))], \quad (22)$$

and ∂^{-2} denotes the inverse Laplacian.

In the present model, the slow-roll parameter ϵ is taken to be always small after and before the transition, this means inflation never stops. However both η and η' can become large temporarily after the transition. Therefore the dominant contribution to the bispectrum is expected to come from the following terms in the action,

$$S_3 \supset M_{Pl}^2 \int dt d^3 x \left[\frac{a^3 \epsilon}{2} \frac{d\eta}{dt} \mathcal{R}^2 \dot{\mathcal{R}} + 2 \frac{\eta}{4} \mathcal{R}^2 \left. \frac{\delta L}{\delta \mathcal{R}} \right|_1 \right] + M_{Pl}^2 \int dt d^3 x \frac{d}{dt} \left[-\frac{\eta a}{2} \mathcal{R}^2 \partial^2 \chi \right], \quad (23)$$

where the last term is a total time derivative but it is retained because it is proportional to η . This is one of the many terms that appear when simplifying the action to the form (20) via integrations by parts [97, 98], which can be important and cannot a priori be neglected [98]. After integration by parts the cubic action can be simplified to

$$S_3 \supset M_{Pl}^2 \int dt d^3 x \left[-\epsilon \eta a^3 \mathcal{R} \dot{\mathcal{R}}^2 - \frac{\epsilon \eta}{2} a \mathcal{R}^2 \partial^2 \mathcal{R} \right], \quad (24)$$

from where one easily finds the cubic-order interaction Hamiltonian as

$$H_{int}(\tau) = M_{Pl}^2 \int d^3 x \left[\epsilon \eta a \mathcal{R} \mathcal{R}'^2 + \frac{\epsilon \eta}{2} a \mathcal{R}^2 \partial^2 \mathcal{R} \right]. \quad (25)$$

The tree-level three-point correlation function (or bispectrum) at the time τ_e after horizon exit is

$$\langle \Omega | \hat{\mathcal{R}}(\tau_e, \mathbf{k}_1) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_2) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_3) | \Omega \rangle = -i \int_{-\infty}^{\tau_e} d\tau a \langle 0 | [\hat{\mathcal{R}}(\tau_e, \mathbf{k}_1) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_2) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_3), \hat{H}_{int}(\tau)] | 0 \rangle. \quad (26)$$

More explicitly, the bispectrum is

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}_1) \hat{\mathcal{R}}(0, \mathbf{k}_2) \hat{\mathcal{R}}(0, \mathbf{k}_3) | \Omega \rangle \approx & (2\pi)^3 \delta^{(3)}(\mathbf{K}) 2M_{Pl}^2 \Im \left[\mathcal{R}(0, \mathbf{k}_1) \mathcal{R}(0, \mathbf{k}_2) \mathcal{R}(0, \mathbf{k}_3) \int_{-\infty}^0 d\tau \eta \epsilon a^2 \mathcal{R}^*(\tau, \mathbf{k}_1) \times \right. \\ & \left. (2\mathcal{R}'^*(\tau, \mathbf{k}_2) \mathcal{R}'^*(\tau, \mathbf{k}_3) - k_1^2 \mathcal{R}^*(\tau, \mathbf{k}_2) \mathcal{R}^*(\tau, \mathbf{k}_3)) \right] + \text{two perms.}, \end{aligned} \quad (27)$$

where $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$ and “two perms.” denotes two additional permutations of the term displayed, and in the rest of this work we set $\tau_e \approx 0$.

Because the integrand of the previous expression is different before and after the transition, here we compute the contribution before the transition, i.e. the integral from $-\infty$ to $\tau_0 = -1/k_0$, separately from the contribution after the transition, i.e. the integral from τ_0 to zero. Obviously the final answer for the bispectrum is the sum of these two contributions. As we show below, for the parameter choice described in the previous section, the contribution after the transition is much larger than the contribution before the transition, so the later one can be ignored.

The contribution before the transition (i.e. the integral is performed from $-\infty$ to τ_0 only) is

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(\tau_e, \mathbf{k}_1) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_2) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_3) | \Omega \rangle_+ = & -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{\eta_+ H_0^4}{32 M_{Pl}^4 \sqrt{\epsilon_+ \epsilon_-^3(\tau_e)}} \frac{1}{(k_1 k_2 k_3)^3} \\ & \times \Im \left[(\alpha_{k_1} - \beta_{k_1}) (\alpha_{k_2} - \beta_{k_2}) (\alpha_{k_3} - \beta_{k_3}) \left(\frac{2i k_2^2 k_3^2}{k_t} e^{-i \frac{k_t}{k_0}} \left(1 + \frac{k_1}{k_t} + i \frac{k_1}{k_0} \right) \right. \right. \\ & \left. \left. + k_1^2 e^{-i \frac{k_t}{k_0}} \left(k_0 - \frac{k_1 k_2 k_3}{k_t k_0} + i \frac{k_1 k_2 k_3}{k_t^2} + i \frac{k_1 k_2 + k_2 k_3 + k_1 k_3}{k_t} \right) \right) \right] + \text{two perms.}, \end{aligned} \quad (28)$$

where $k_t \equiv k_1 + k_2 + k_3$. In the equilateral limit and after using the explicit expressions for the slow-roll parameters it simplifies to

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_+ = -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{27}{4} \frac{H_0^8 A_+}{M_{Pl}^2 A_-^3} \frac{1}{k^6} \Im \left[(\alpha_k - \beta_k)^3 e^{-3i \frac{k}{k_0}} \left(2i - \frac{k}{k_0} + \frac{k_0}{k} \right) \right]. \quad (29)$$

In the large scales limit, i.e. $k \ll k_0$, it further simplifies to

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_+ = (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{27}{4} \frac{H_0^8}{M_{Pl}^2 A_+^2} \frac{1}{k^6} + \mathcal{O}\left(\frac{1}{k_0^2 k^4}\right). \quad (30)$$

The previous bispectrum is at leading order proportional to $1/k^6$. This is the scaling expected for a scale invariant model.

On small scales, i.e. when $k_0 \ll k$, we get

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_+ &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{27}{4} \frac{H_0^8 A_+}{M_{Pl}^2 A_-^3} \frac{1}{k^6} \\ &\times \left[\frac{k}{k_0} \sin\left(\frac{3k}{k_0}\right) + \left(2 - \frac{9\Delta A}{2A_+}\right) \cos\left(\frac{3k}{k_0}\right) - \frac{9\Delta A}{2A_+} \cos\left(\frac{k}{k_0}\right) + \mathcal{O}\left(\frac{k_0}{k}\right) \right], \end{aligned} \quad (31)$$

where the first term in square brackets is the leading one and the others are the sub-leading corrections, one of which oscillates with a different “angular frequency”. As one can see, in this limit the bispectrum scales as $1/k^5$. This represents a linear growth envelope with respect to the bispectrum of a scale-invariant model. The “angular frequency” of the leading-order term is set by the transition scale and is $3/k_0$.

The calculation of the *contribution after the transition* (i.e. the integral from τ_0 to τ_e) is more involved but can be done analytically. The full result can be found in Appendix A. In the equilateral limit it simplifies to

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_- &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^6}{2} \frac{H_0^{12}}{A_-^4} \frac{1}{k^9} \\ &\times \Re \left[(\alpha_k - \beta_k)^3 \left((\alpha_k^*)^3 F_1 + (\beta_k^*)^3 F_1^* + (\alpha_k^*)^2 \beta_k^* F_2 + \alpha_k^* (\beta_k^*)^2 F_2^* \right) \right], \end{aligned} \quad (32)$$

where the functions F_1 and F_2 are defined as (for $\tau_e = 0$)

$$F_1 = \frac{\rho^3 i}{2} \left[-1 - e^{-3i \frac{k}{k_0}} \frac{\left(i - \frac{k}{k_0}\right)^2}{\left(1 + \frac{\rho^3}{k_0^3}\right)^3} \left(1 + i \frac{k}{k_0} - \left(\frac{k}{k_0}\right)^2 + 3 \left(\frac{\rho}{k_0}\right)^3 + i \frac{3k\rho^3}{k_0^4} - \frac{\rho^3 k^2}{k_0^5} \right) \right], \quad (33)$$

$$F_2 = \frac{\rho^3 i}{2} \left[3 - e^{-i \frac{k}{k_0}} \frac{\left(i - \frac{k}{k_0}\right)}{\left(1 + \frac{\rho^3}{k_0^3}\right)^3} \left(-3i + \left(\frac{k}{k_0}\right)^3 - i \frac{9\rho^3}{k_0^3} - i \frac{6k^2 \rho^3}{k_0^5} + \frac{k^3 \rho^3}{k_0^6} \right) \right], \quad (34)$$

where $\rho^3 = -(\Delta A/A) - k_0^3$. Equation (32) is one of the main results of this paper.

On large scales, i.e. $k \ll k_0$, it simplifies to

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_- &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^7}{20} \frac{H_0^{12} \Delta A}{A_- A_+^4} \frac{1}{k^6} \\ &\times \left[\left(\frac{k}{k_0}\right)^2 + \left(\frac{k}{k_0}\right)^4 \left(\frac{41}{70} - \frac{4A_+}{5A_-}\right) + \left(\frac{k}{k_0}\right)^6 \left(\frac{-38851}{113400} - \frac{4246A_+}{14175A_-} + \frac{4A_+^2}{25A_-^2}\right) \right. \\ &\quad \left. + \left(\frac{k}{k_0}\right)^8 \left(\frac{225409}{3492720} + \frac{5701A_+}{39690A_-} - \frac{31A_+^2}{2835A_-^2}\right) + \mathcal{O}\left(\frac{k}{k_0}\right)^{10} \right], \end{aligned} \quad (35)$$

where the first term in square brackets is the leading one and we have included several other sub-leading terms because they will be used to compare our result with previously known results available in the literature. On large scales the quantity $k^6 \times$ bispectrum is proportional to $\frac{k^2}{k_0^2}$ so its amplitude decreases for larger and larger scales until this contribution becomes smaller than the contributions coming from the slow-roll neglected terms in the previous calculation. These contributions are expected to be small and perhaps unobservable. Some (but not all) of these contributions were computed in [92] and they turn out to be of the scale-invariant kind (that is the quantity $k^6 \times$ bispectrum is independent of k).

On small scales, i.e. when $k_0 \ll k$, we get

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_- &= (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^6}{4} \frac{H_0^{12} \Delta A}{A_+^2 A_-^3} \frac{1}{k^6} \\ &\times \left[\frac{k}{k_0} \sin\left(\frac{3k}{k_0}\right) + \left(\frac{A_-}{A_+} + 2 - \frac{9\Delta A}{2A_+}\right) \cos\left(\frac{3k}{k_0}\right) - \frac{9\Delta A}{2A_+} \cos\left(\frac{k}{k_0}\right) + \mathcal{O}\left(\frac{k_0}{k}\right) \right]. \end{aligned} \quad (36)$$

In this limit, the leading order term (first term inside the square brackets) is a linear growth envelope with oscillations of an “angular frequency” $3/k_0$. This linear growth may potentially have important implications regarding the detectability of this signal. This result is similar to the one found for instance in Ref. [96]. Also for the parameters values choice of the previous section and used in [92] one can show that the contribution (36) to the integral dominates over Eq. (31). In fact the ratio of these amplitudes is proportional to $\Delta A/(A_+ \epsilon_+)$ which is always large for the case of interest. Similar result holds on large scales, where Eq. (35) dominates over Eq. (30) (this is true for scales larger than the transition but not much larger, because Eq. (35) decays quickly as the length scale increases).

Finally, it is instructive to estimate the range of wavenumbers Δk over which large deviations from Gaussianity maybe be expected in this model. If the sharp transition of the Starobinsky model (1) is replaced with a smooth transition with a width in field space $\Delta\phi$ then the number of e -foldings that takes to cross the transition is $\Delta N \sim H_0/\dot{\phi}\Delta\phi \sim -V_0/(M_{Pl}^2 A_-)\Delta\phi$. Because $\Delta k/k_0 \sim 1/|\Delta N|$ one finds $\Delta k/k_0 \sim (M_{Pl}^2 A_-)/(\Delta\phi V_0)$. In the present model $\Delta\phi$ is zero which implies that the range of scales affected is $\Delta k \rightarrow \infty$. In a more realistic model, we expect that any transition is eventually smooth in a sufficient small scale, this transition width will introduce a cut-off for the growth of the bispectrum as described by the previous equation. For scales smaller than this cut-off scale the previous equation is not valid, instead we expect the amplitude of the deviations from Gaussianity to go quickly to zero. This expectation was recently confirmed in a different but somehow related model (the potential has a step-like feature) [91].

B. Comparison with previous works

As mentioned in the introduction the bispectrum in this model was first computed in [70]. There, the spatial gradient expansion approach was used at next-to-leading order. Because of this their result is expected to be correct only on large scales and also in the limit $A_+ \gg A_-$. In the equilateral limit, the first equation in Eq. (6.20) of [70] is

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^7}{10} \frac{H_0^{12} \Delta A}{A_- A_+^4} \frac{1}{k^6} \\ &\times \left[\left(\frac{k}{k_0}\right)^2 + \frac{4}{5} \left(\frac{k}{k_0}\right)^4 \left(1 - \frac{A_+}{A_-}\right) + \frac{4}{25} \left(\frac{k}{k_0}\right)^6 \left(1 - 2\frac{A_+}{A_-} + \frac{A_+^2}{A_-^2}\right) + \frac{1}{9} \left(\frac{k}{k_0}\right)^8 \left(1 - 2\frac{A_+}{A_-} + \frac{A_+^2}{A_-^2}\right) \right]. \end{aligned} \quad (37)$$

Using the fact that $A_+ \gg A_-$ the above equation is, up to a factor of 2, equal to Eq. (35) at leading order but it differs in the term proportional to $(k/k_0)^8$. This extra factor of 2 in Eq. (37) is due to a double counting mistake of the permutations in [70]¹. We have not found a reason for the discrepancy in the term proportional to $(k/k_0)^8$. Finally one can compare the result of [70] for a general triangular configuration with Eq. (A1) of Appendix A. Correcting the factor of 2 difference in the overall amplitude due to the double counting mistake, we again find that their result agrees with ours up to a discrepancy in the sub-leading term of the order of k^2/k_0^8 .

The second attempt [92] to compute the bispectrum (equilateral configuration only) for this model used the in-in formalism just like in the present paper and even computed analytically some of the sub-leading order corrections (although not all of them). They found that the dominant bispectrum contribution on both large and small scales goes as $1/k^6$ and for small scales the amplitude oscillates with $\cos\left(\frac{3k}{k_0}\right)$. As discussed in the previous subsection we found a different result. On large scale the bispectrum scales as $1/(k_0^2 k^4)$ while on small scales it possesses a linear growth envelope as $1/(k_0 k^5) \sin(3k/k_0)$. We discuss the origin of this difference below, but in short it appears that Ref. [92] missed a term in the calculation that is proportional to the Dirac delta function.

¹ We thank Yuichi Takamizu for discussions on this point.

In Ref. [92] an alternative form of the action² given in Eq. (23) is used. Keeping the term proportional to $\epsilon\eta'$ only one finds

$$\langle \Omega | \hat{\mathcal{R}}(\tau_e, \mathbf{k}_1) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_2) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_3) | \Omega \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{K}) i M_{Pl}^2 \mathcal{R}(\tau_e, \mathbf{k}_1) \mathcal{R}(\tau_e, \mathbf{k}_2) \mathcal{R}(\tau_e, \mathbf{k}_3) \int_{-\infty}^{\tau_e} d\tau a^2 \epsilon \eta' (\mathcal{R}^*(\tau, \mathbf{k}_1) \mathcal{R}^*(\tau, \mathbf{k}_2) \mathcal{R}^{*'}(\tau, \mathbf{k}_3) + \text{two perms.}) + \text{c.c.}, \quad (38)$$

where “c.c.” denotes the complex conjugate. The action (23) contains two further terms, one proportional to the first order equations of motion, but this term gives an exactly zero contribution when evaluated on-shell. The second term is a boundary term. Which should be evaluated at the past infinity boundary, this gives zero once we rotate the contour into the imaginary plane to select the free vacuum. And it should also be evaluated at a later time (a time when all scales of interest are well outside the horizon). However at such a late time after the transition, slow-roll will have been restored and again the contribution from this term is sub-dominant. Therefore one concludes that Eq. (38) contains the dominant contribution.

Using the Klein-Gordon equation, $\dot{\eta}$ can be written as

$$\dot{\eta} = -\frac{2}{H} \frac{d^2 V}{d\phi^2} + \dots = -\frac{2}{H} \frac{A_+ - A_-}{|\phi'(\tau_0)|} \delta(\tau - \tau_0) + \dots, \quad (39)$$

where \dots denotes other terms without Dirac delta functions, the last equality is valid for the Starobinsky model and we have used $\delta(\phi_0 - \phi) = \delta(\tau - \tau_0)/|\phi'(\tau_0)|$. If we keep only this term in Eq. (38) the integral simplifies to

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(\tau_e, \mathbf{k}_1) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_2) \hat{\mathcal{R}}(\tau_e, \mathbf{k}_3) | \Omega \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{4M_{Pl}^2 \epsilon(\tau_0) (A_+ - A_-) a^3(\tau_0)}{|\phi'(\tau_0)| H(\tau_0)} \\ &\quad \times \Im [\mathcal{R}(\tau_e, \mathbf{k}_1) \mathcal{R}(\tau_e, \mathbf{k}_2) \mathcal{R}(\tau_e, \mathbf{k}_3) \mathcal{R}^*(\tau_0, \mathbf{k}_1) \mathcal{R}^*(\tau_0, \mathbf{k}_2) \mathcal{R}^{*'}(\tau_0, \mathbf{k}_3)] \\ &\quad + \text{two permutations.} \end{aligned} \quad (40)$$

This contribution dominates over the dominant contribution calculated in [92] on small scales and is of the same order of magnitude for large scales.

For the equilateral configuration, i.e. $k_1 = k_2 = k_3 = k$, the bispectrum (40) with $\tau_e = 0$ simplifies to

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{K}) \left(-\frac{3^6}{4} \right) \frac{\Delta A}{A_+^2 A_-^3} \frac{H_0^{12}}{k^5 k_0} \Im \left[(\alpha_k - \beta_k)^3 \left(1 - i \frac{k_0}{k} \right)^2 e^{-3i \frac{k}{k_0}} \right]. \quad (41)$$

In the large scale limit, i.e. $k \ll k_0$, this reads

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{K}) \left(-\frac{3^6}{4} \right) \frac{\Delta A}{A_+^5} \frac{H_0^{12}}{k^6} \left[1 + \left(\frac{k}{k_0} \right)^2 \frac{17A_- - 2A_+}{10A_-} + \mathcal{O} \left(\frac{k}{k_0} \right)^4 \right]. \quad (42)$$

On small scales, i.e. $k_0 \ll k$ one finds

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^6}{4} \frac{\Delta A}{A_+^2 A_-^3} \frac{H_0^{12}}{k^6} \\ &\quad \times \left[\frac{k}{k_0} \sin \left(\frac{3k}{k_0} \right) + \left(2 - \frac{9\Delta A}{2A_+} \right) \cos \left(\frac{3k}{k_0} \right) - \frac{9\Delta A}{2A_+} \cos \left(\frac{k}{k_0} \right) + \mathcal{O} \left(\frac{k_0}{k} \right) \right]. \end{aligned} \quad (43)$$

If one calculates $\mathcal{G}/k^3 \sim k^6 \times \text{bispectrum}$ (the precise definition of \mathcal{G} is given in the next subsection) then on small scales one finds that $\mathcal{G}/k^3 \sim \frac{k}{k_0} \sin \left(\frac{3k}{k_0} \right)$ which is the linear growth found in [96] and in the previous subsection. On large scales $\mathcal{G}/k^3 \sim \text{constant}$. In order to obtain the final answer for the bispectrum one needs to add to the above the contribution from all the other terms that was first calculated in Ref. [92].

² One is free to use this form of the action or the form used in the previous subsection or possibly others, the end result is the same as we shall show below and the two procedures are equivalent.

Using their Eqs. (100)-(104) one has that the contribution is

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^7}{4} \frac{\Delta A}{A_-^5} \frac{H_0^{12}}{k^6} \left(\frac{k_0}{k} \right)^3 \\ &\times \left[\mathcal{A}_1(k) \sin \left(\frac{k}{k_0} \right) + \mathcal{A}_2(k) \cos \left(\frac{k}{k_0} \right) + \mathcal{A}_3(k) \sin \left(\frac{3k}{k_0} \right) + \mathcal{A}_4(k) \cos \left(\frac{3k}{k_0} \right) \right], \end{aligned} \quad (44)$$

where

$$\mathcal{A}_1(k) = \frac{3A_-^3 \Delta A}{4A_+^5} \left(1 + \frac{k^2}{k_0^2} \right)^2 \left[9A_- \left(1 + \frac{k^2}{k_0^2} \right) + A_+ \left(-9 - 9\frac{k^2}{k_0^2} + 2\frac{k^4}{k_0^4} \right) \right] \left(\frac{k_0}{k} \right)^6, \quad (45)$$

$$\mathcal{A}_2(k) = -\frac{3A_-^3 \Delta A}{4A_+^5} \left(1 + \frac{k^2}{k_0^2} \right)^2 \left[9A_- \left(1 + \frac{k^2}{k_0^2} \right) - A_+ \left(9 + 11\frac{k^2}{k_0^2} \right) \right] \left(\frac{k_0}{k} \right)^5, \quad (46)$$

$$\begin{aligned} \mathcal{A}_3(k) &= -\frac{A_-^3}{12A_+^5} \left[27(\Delta A)^2 \left(1 - \frac{k^2}{k_0^2} \right) - 27\Delta A (5A_- - 7A_+) \frac{k^4}{k_0^4} - (9A_- - 11A_+)^2 \frac{k^6}{k_0^6} \right. \\ &\quad \left. + 6A_+ (-3A_- + 5A_+) \frac{k^8}{k_0^8} \right] \left(\frac{k_0}{k} \right)^6, \end{aligned} \quad (47)$$

$$\begin{aligned} \mathcal{A}_4(k) &= \frac{A_-^3}{12A_+^5} \left[-27A_-^2 \left(-3 + \frac{k^2}{k_0^2} \right) \left(1 + \frac{k^2}{k_0^2} \right)^2 + 18A_- A_+ \left(1 + \frac{k^2}{k_0^2} \right) \left(-9 - 7\frac{k^2}{k_0^2} + 6\frac{k^4}{k_0^4} \right) \right. \\ &\quad \left. + A_+^2 \left(81 + 153\frac{k^2}{k_0^2} - 9\frac{k^4}{k_0^4} - 93\frac{k^6}{k_0^6} + 4\frac{k^8}{k_0^8} \right) \right] \left(\frac{k_0}{k} \right)^5. \end{aligned} \quad (48)$$

On large scales, i.e. $k \ll k_0$, one finds that the bispectrum (44) reads

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^6}{4} \frac{\Delta A}{A_+^5} \frac{H_0^{12}}{k^6} \left[1 + \left(\frac{k}{k_0} \right)^2 \frac{17A_- - 8A_+}{10A_-} + \mathcal{O} \left(\frac{k}{k_0} \right)^4 \right]. \quad (49)$$

On small scales, i.e. $k_0 \ll k$ one finds

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^6}{4} \frac{\Delta A}{A_+^3 A_-^2} \frac{H_0^{12}}{k^6} \left[\cos \left(\frac{3k}{k_0} \right) + \mathcal{O} \left(\frac{k_0}{k} \right) \right]. \quad (50)$$

Then it can be seen that the sum of Eq. (44) with the Dirac delta function contribution given by Eq. (41) exactly agrees the result in Eq. (32) given in the previous subsection. This explains the discrepancy between our result and the result of [92] (it also explains the previously existing discrepancy between the large scales results of [92] and [70]). The point is that the Dirac delta function term in $\dot{\eta}$ of Eq. (39) cannot be neglected because on large scales it gives a contribution comparable to Eq. (49) and on small scales it actually dominates over Eq. (50). The plots of these different contributions for the final bispectrum are given in Fig. 1, with the choice of model parameters described in Section II.

C. The non-linearity function $\mathcal{G}(k)/k^3$

In this subsection, in order to gain a feeling for the shape and size of the bispectrum in the equilateral limit for the Starobinsky model, we plot the previous results for the choice of parameters as in Section II.

To show the shape of the bispectrum when the three wavenumbers are of comparable size one commonly plots a quantity defined by

$$\frac{\mathcal{G}(k_1, k_2, k_3)}{k_1 k_2 k_3} = \frac{1}{\delta^{(3)}(\mathbf{K})} \frac{(k_1 k_2 k_3)^2}{(2\pi)^7 P_{\mathcal{R}}^2} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}_1) \hat{\mathcal{R}}(0, \mathbf{k}_2) \hat{\mathcal{R}}(0, \mathbf{k}_3) | \Omega \rangle. \quad (51)$$

If this quantity is of order unity there may be some hope that it will be measured in the near future for example with ESA's Planck satellite. Because in the present model the amplitudes of the power spectrum before and after k_0 can

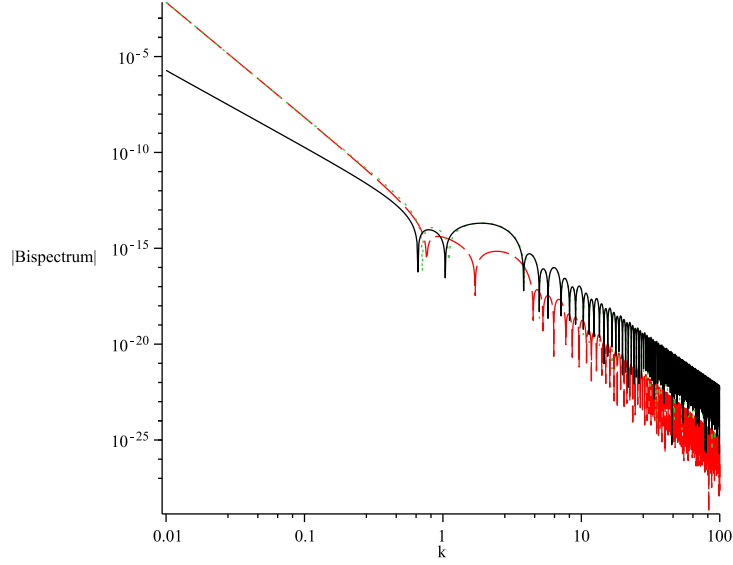


FIG. 1: Plot of the absolute value of the equilateral bispectra for $k_0 = 1$ as a function of k . The red long-dashed line is the plot of Eq. (44), the green dotted line in the plot of the Dirac delta function contribution to the bispectrum as given by Eq. (41) and the solid black line is the plot of Eq. (32). One can see that for large scales the contributions from Eq. (44) and Eq. (41) are comparable in size and because they have opposite signs they partially cancel to give the solid black line. On small scales the amplitude of the long-dashed red line is negligible compared with the dotted green line. For this scales the dotted green line and the solid black line are virtually identical. The model parameters have the values described in Sec. II.

be quite different it makes more sense to define two functions $\mathcal{G}_<$ and $\mathcal{G}_>$ to study the bispectrum for scales larger and smaller than the transition scale k_0 respectively, as

$$\frac{\mathcal{G}_<(k_1, k_2, k_3)}{k_1 k_2 k_3} = \frac{1}{\delta^{(3)}(\mathbf{K})} \frac{(k_1 k_2 k_3)^2}{(2\pi)^7 P_<^2} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}_1) \hat{\mathcal{R}}(0, \mathbf{k}_2) \hat{\mathcal{R}}(0, \mathbf{k}_3) | \Omega \rangle_-, \quad (52)$$

$$\frac{\mathcal{G}_>(k_1, k_2, k_3)}{k_1 k_2 k_3} = \frac{1}{\delta^{(3)}(\mathbf{K})} \frac{(k_1 k_2 k_3)^2}{(2\pi)^7 P_>^2} \langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}_1) \hat{\mathcal{R}}(0, \mathbf{k}_2) \hat{\mathcal{R}}(0, \mathbf{k}_3) | \Omega \rangle_-, \quad (53)$$

where $P_<$ and $P_>$ denote the asymptotic values of the power spectrum from large and small scales respectively, and are given by

$$P_< = \left(\frac{H_0}{2\pi} \right)^2 \left(\frac{3H_0^2}{A_+} \right)^2, \quad (54)$$

$$P_> = \left(\frac{H_0}{2\pi} \right)^2 \left(\frac{3H_0^2}{A_-} \right)^2. \quad (55)$$

With this last definition we make sure that the oscillations seen in the quantity $\mathcal{G}_>$ are truly bispectrum oscillations and are not due to the oscillations already present in the small scales power spectrum.

As one can see from Fig. 2, on large scales and with our choice of parameters of Sec. II the amplitude of $\mathcal{G}_</math> is small and decays quickly towards larger scales. At some point this amplitude becomes so small that it is subdominant with respect to the slow-roll correction ignored in this work. Some of these slow-roll corrections were computed in [92]. On the other hand, Fig. 3 shows a very interesting behaviour for the non-linearity parameter $\mathcal{G}_>/k^3$. One finds rapid oscillations with a growing amplitude towards smaller scales. For the model parameter of Sec. II the amplitude can easily become much larger than one.$

IV. CONCLUSION

In this work, we revisited the computation of the tree-level three-point function (or bispectrum) in the Starobinsky model of inflation. In this model, a vacuum energy dominated potential is supplemented with a linear term in which

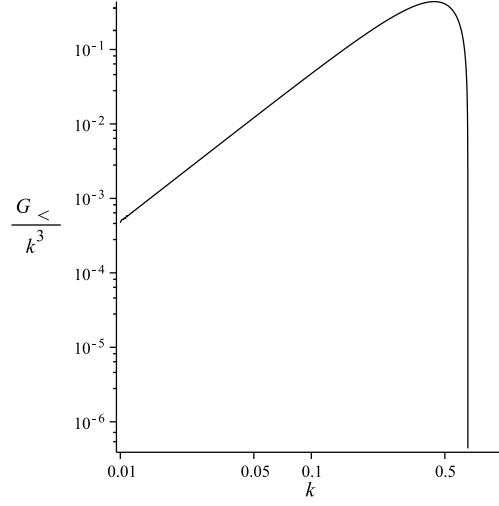


FIG. 2: Plot of $\mathcal{G}_{<}/k^3$ as a function of k for $k_0 = 1$ and the model parameters of Sec. II. On large scales one finds that the amplitude of $\mathcal{G}_{<}/k^3$ is much smaller than one and is decreasing for larger scales. This is an expected result as for these scales the transition happened when they were super-horizon, because \mathcal{R} is non-linearly conserved the deviation from Gaussianity should be small as these perturbations were nearly Gaussian at horizon crossing. For very large scales, eventually the magnitude of this contribution becomes smaller than the always present slow-roll corrections that we neglected in this work. When this happens these sub-leading corrections become the dominant part. However they are slow-roll suppressed and so less interesting observationally.

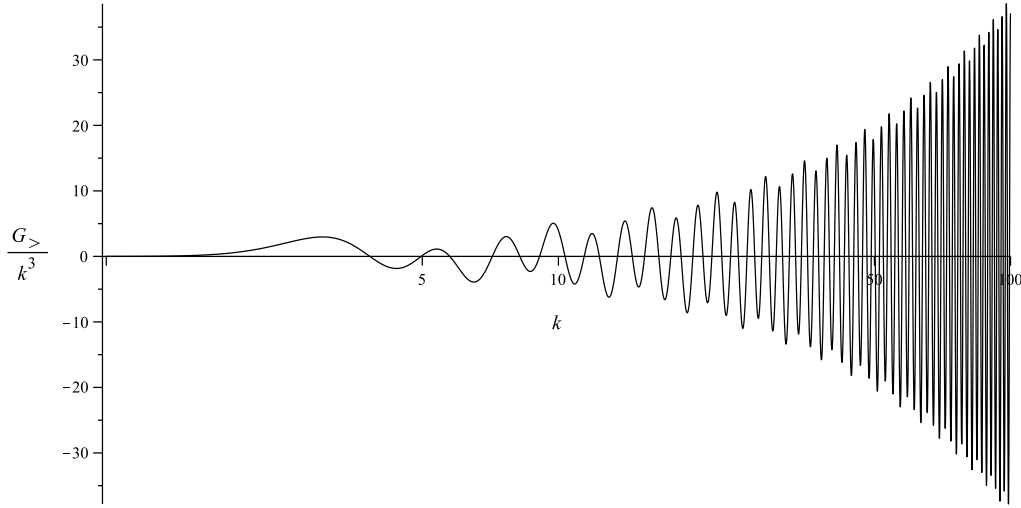


FIG. 3: Plot $\mathcal{G}_{>}/k^3$ as a function of k for $k_0 = 1$ and the model parameters of Sec. II. On small scales the amplitude of the analogue of the non-linearity parameter, i.e. $\mathcal{G}_{>}/k^3$, oscillates with an “angular frequency” given by $3/k_0$, where k_0 is the transition scale and the envelope of the oscillations grows linearly with k/k_0 . One easily sees that the amplitude of $\mathcal{G}_{>}/k^3$ can reach values much larger than one and be of potential interest observationally.

the slope changes abruptly at a point. It is well known that under the assumption of vacuum energy domination this model admits an analytical solution not only for the background equations of motion but also for the mode functions. Thus despite the fact that the slow-roll approximation is broken for a period of time just after the transition happens this model still admits an accurate analytical approximation for the mode functions of linear perturbation theory.

Using this analytical approximation for the mode function we computed analytically the dominant contribution to the bispectrum. The main result is Eq. (32). In the equilateral limit, we obtained that on scales larger than the scale of the transition k_0 the dominant contribution to the bispectrum scales as $1/(k_0^2 k^4)$, that is, the analogue of the non-linearity parameter, \mathcal{G}/k^3 , scales as $\mathcal{G}/k^3 \sim (k/k_0)^2$. This is in agreement with the leading-order result of [70] (after correcting its factor two error) but it differs from the large scale result of [92]. This last result was found to scale

as $\mathcal{G}/k^3 \sim \text{constant}$. We explained the origin of this difference with the fact that Ref. [92] missed the contribution coming from a term in the cubic-order action that contains the Dirac delta function in the time derivative of the coupling constant, $\epsilon\dot{\eta}$. This missing contribution has the opposite sign to the dominant term they calculated and it partially cancels it out and one is left with the result we found, that is $\mathcal{G}/k^3 \sim (k/k_0)^2$. For small scales, we found some interesting behaviour. The parameter \mathcal{G}/k^3 scales as $\mathcal{G}/k^3 \sim k/k_0 \sin(3k/k_0)$. The “angular frequency” of the small scale oscillations is set by the transition scale and is given by $3/k_0$. The envelope function of these oscillations in a linear growth toward small scales. A similar result was found in [96] for a somehow related model. This linear growth represents a strong scale dependence of the non-linearity parameter in this model and the k/k_0 enhancement factor can have important consequences regarding observations. It would be interesting to consider observational constraints on this kind of oscillating and growing amplitude \mathcal{G}/k^3 models, however that study is beyond the scope of the present work. Once again this result differs from the result found in Ref. [92]. There, it was found that \mathcal{G}/k^3 scales as $\mathcal{G}/k^3 \sim \cos(3k/k_0)$, with no enhancement factor. The reason for the difference with respect to our result is again the fact that the contribution from the Dirac delta function was missed. Finally, we briefly showed that the amplitude of \mathcal{G}/k^3 can reach values much larger than one on small scales. The large enhancement factor k/k_0 is key in reaching that conclusion.

For future work, we leave the computation of the trispectrum or four-point function.

Acknowledgments

We would like to thank Yuichi Takamizu for discussions regarding the comparison of our results with the result of Takamizu *et al.* [70]. FA acknowledges the support by the World Class University grant no. R32-10130 through the National Research Foundation, Ministry of Education, Science and Technology of Korea. MS is supported in part by JSPS Grant-in-Aid for Scientific Research (A) No. 21244033. This work was also supported in part by MEXT Grant-in-Aid for the global COE program at Kyoto University, “The Next Generation of Physics, Spun from Universality and Emergence”.

Appendix A: The bispectrum for any configuration of the triangle

In this Appendix we will present the result for the bispectrum that is valid for any allowed configuration of the three momentum vectors.

As discussed in the main text, the integral in Eq. (27) can be split in a contribution before the transition and a contribution after the transition. For a general triangle configuration, the contribution before the transition is given by Eq. (28) and it is sub-leading with respect to the contribution after the transition. Therefore for the scales of interest the bispectrum is well approximated by the contribution after the transition only. It turns out that the integral after the transition can also be evaluated analytically. After a lengthy but straightforward calculation, the result for the bispectrum is

$$\begin{aligned} \langle \Omega | \hat{\mathcal{R}}(\tau_e, \mathbf{k}) \hat{\mathcal{R}}(\tau_e, \mathbf{k}) \hat{\mathcal{R}}(\tau_e, \mathbf{k}) | \Omega \rangle_- &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{9H_0^6}{8\sqrt{2}A_- M_{Pl}^3 \sqrt{\epsilon_-^3(\tau_e)}} \frac{1}{(k_1 k_2 k_3)^3} \\ &\times \Re \left[(\alpha_{k_1} - \beta_{k_1})(\alpha_{k_2} - \beta_{k_2})(\alpha_{k_3} - \beta_{k_3}) \left(\alpha_{k_1}^* \alpha_{k_2}^* \alpha_{k_3}^* T(k_1, k_2, k_3) - \beta_{k_1}^* \beta_{k_2}^* \beta_{k_3}^* T(-k_1, -k_2, -k_3) \right. \right. \\ &\quad - \alpha_{k_1}^* \alpha_{k_2}^* \beta_{k_3}^* T(k_1, k_2, -k_3) - \alpha_{k_1}^* \alpha_{k_3}^* \beta_{k_2}^* T(k_1, -k_2, k_3) - \alpha_{k_2}^* \alpha_{k_3}^* \beta_{k_1}^* T(-k_1, k_2, k_3) \\ &\quad \left. \left. + \alpha_{k_1}^* \beta_{k_2}^* \beta_{k_3}^* T(k_1, -k_2, -k_3) + \alpha_{k_2}^* \beta_{k_3}^* \beta_{k_1}^* T(-k_1, k_2, -k_3) + \alpha_{k_3}^* \beta_{k_1}^* \beta_{k_2}^* T(-k_1, -k_2, k_3) \right) \right], \quad (\text{A1}) \end{aligned}$$

where $T(k_1, k_2, k_3)$ is defined as

$$\begin{aligned} T(k_1, k_2, k_3) &= \left[\frac{i\rho^3}{(1 - \rho^3 \tau^3)^3} e^{i\tau k_t} \left(-3 + k_1 k_2 k_3 k_t \rho^3 \tau^7 + i\rho^3 \tau^6 (9k_1 k_2 k_3 + k_1(k_2^2 + k_3^2) + k_2(k_1^2 + k_3^2) + k_3(k_1^2 + k_2^2)) \right. \right. \\ &\quad - \rho^3 \tau^5 (k_1^2 + k_2^2 + k_3^2 + 9(k_1 k_2 + k_2 k_3 + k_3 k_1)) - \tau^4 (k_1 k_2 k_3 k_t + 9i\rho^3 k_t) \\ &\quad \left. \left. + \tau^3 (9\rho^3 - ik_t(k_1 k_2 + k_2 k_3 + k_3 k_1)) + \tau^2 (k_1^2 + k_2^2 + k_3^2 + 3(k_1 k_2 + k_2 k_3 + k_3 k_1)) + 3ik_t \tau \right) \right] \Big|_{-k_0^{-1}}^{\tau_e}. \quad (\text{A2}) \end{aligned}$$

For $\tau_e = 0$, the function $T(k_1, k_2, k_3)$ is related to the functions F_1 and F_2 defined in the main text as $F_1 = T(k, k, k)/6$ and $F_2 = -T(k, k, -k)/2$ and in the equilateral configuration Eq. (A1) reduces to Eq. (32).

Finally it is worth noting that in the squeezed limit of the triangle formed by the three momenta, i.e. when $k_2 \sim k_3 \sim k$ and the small momentum k_1 is $k_1 \ll k$, one can find on large scales, i.e. when $k_0 \gg k \gg k_1$,

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_- \approx -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{162}{5} \frac{H_0^{12} \Delta A}{A_- A_+^4} \frac{1}{k_1^3 k^3} \left(\frac{k}{k_0} \right)^2, \quad (\text{A3})$$

and on small scales, i.e. when $k_0 \ll k_1 \ll k$, one finds

$$\langle \Omega | \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) \hat{\mathcal{R}}(0, \mathbf{k}) | \Omega \rangle_- \approx (2\pi)^3 \delta^{(3)}(\mathbf{K}) \frac{3^5}{2} \frac{H_0^{12} \Delta A}{A_-^3 A_+^2} \frac{1}{k_1^3 k^3} \left(\frac{k_1}{k_0} \sin \left(\frac{2k}{k_0} \right) \right), \quad (\text{A4})$$

where the enhancement factor k_1/k_0 in the previous equation is $k_1/k_0 \gg 1$.

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